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Certain properties of Berry's phases in supersymmetric quantum mechanics

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Abstract. We discuss certain properties of Berry's phases in two quantum systems with supersymmetrically related Hamiltonians and derive an explicit expression for the difference in their Berry's phases. Moreover, we obtain an expression for a topological quantity which can be interpreted in term of holonomy, and which contains the Berry's phase and is invariant in the two supersymmetrically related quantum systems. We take two examples to illustrate our findings. The first one is the example of Aharonov–Bohm effect in which the Berry's phases of the two supersymmetrically related quantum systems are equal. The second example is a system with a spin-$\frac{1}{2}$ particle in a time varying magnetic field. We obtain the stated topological quantity and, in our analysis, we discover that this quantity is corresponding to the essential gauge transformation in such systems.

1. Introduction

Since the advent of the concept of Berry's phase in 1984 [1], a lot of attention has been drawn to the study in both quantum mechanics and classical physics [2]. In fact, the characteristics of Berry's phase can be explained in mathematics as the holonomy in a Hermitian line bundle [3–5], and can be generalized to non-adiabatic (and non-cyclic Hamiltonian) situations [6, 7] and non-Abelian cases [7, 8]. Recently, some fresh developments have emerged: for instance, the study of Berry's phase using the evolution operator approach [9–11], the extension of Berry's phase format to a non-Hermitian system [11], analysis of the Berry's phases in coherent states [12–16] and others associated with the interpretations using features of the Berry's phase [17–21].

Meanwhile, supersymmetric quantum mechanics [22, 23] is another fruitful concept in physics and has been widely studied for the last decade [24–32]. Apart from one-dimensional systems [22], the generalizations to higher-dimensional systems have also been presented [33–36]. In fact, supersymmetric quantum mechanics is quite a general theory in quantum mechanics. It is then interesting to investigate the properties of Berry's phases in supersymmetric quantum mechanics.

Since Berry's phase in general results from cyclic evolution of a quantum system, we need to discuss first certain relevant features of time-dependent supersymmetric quantum mechanics in section 2. In particular, we shall make an assumption (with a concrete example presented in section 5) that the instantaneous positive energy eigenstates of the two supersymmetrically related Hamiltonians can be transformed into each other (equation (2.5)) rendering our following discussions feasible.

We shall then find, as shown in section 3, that the Berry's phases involving two supersymmetrically related eigenstates are not independent. In fact, they are different by a factor which can be evaluated by using the supersymmetric properties.

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Following this, we construct a topological quantity in each of the two supersymmetrically related systems in section 4. We then find out that these two topological quantities are in fact identical. In terms of topological terminologies, we identify the invariant topological quantity with the holonomy corresponding to the connection which is invariant in the two supersymmetrically related quantum systems.

In section 5, we propose two examples to illustrate our results. The first one is the example of Aharonov–Bohm effect [36] similar to that discussed in [1]. It is shown that in this example, the Berry's phases for the two supersymmetrically related quantum systems are identical and also play the role of the invariant topological quantity we obtained in section 4.

The second example is the system of a spin-$\frac{1}{2}$ particle in a magnetic field. We construct the supersymmetric partner of this system and evaluate the Berry's phases, their difference and the invariant topological quantity mentioned in section 4 for these two supersymmetrically related systems. Moreover, apart from the topological interpretation in section 4, the invariant quantity is also shown to correspond to the essential gauge transformation and hence could be related to concrete physical meaning. Section 6 concludes our findings of this paper, together with discussions of certain relevant issues.

2. Time-dependent supersymmetric quantum mechanics

Let us consider a supersymmetric Hamiltonian $\hat{\mathscr{H}}$ which is parametrized by time-dependent parameters $\boldsymbol{R} = (X, Y, \dots)$. At any instant, the supersymmetric Hamiltonian $\hat{\mathscr{H}}(\boldsymbol{R})$ is single valued and can be written in terms of the supercharge operators, $\hat{Q}(\boldsymbol{R})$ and $\hat{Q}^+(\boldsymbol{R})$ [22, 24]:

$$\hat{\mathscr{H}}(\boldsymbol{R}) = \{\hat{Q}(\boldsymbol{R}), \hat{Q}^+(\boldsymbol{R})\} \tag{2.1}$$

or in the diagonal matrix form:

$$\hat{\mathscr{H}}(\boldsymbol{R}) = \begin{pmatrix} \hat{H}_1(\boldsymbol{R}) & 0 \\ 0 & \hat{H}_2(\boldsymbol{R}) \end{pmatrix} \tag{2.2}$$

where $\hat{H}_1(\boldsymbol{R})$ and $\hat{H}_2(\boldsymbol{R})$ are the supersymmetrically related Hamiltonians and can be factorized into the following forms:

$$\hat{H}_1(\boldsymbol{R}) = \hat{A}^+(\boldsymbol{R})\hat{A}^-(\boldsymbol{R}) \qquad \text{and} \qquad \hat{H}_2(\boldsymbol{R}) = \hat{A}^-(\boldsymbol{R})\hat{A}^+(\boldsymbol{R}) \tag{2.3}$$

with any linear operators $\hat{A}^\pm(\boldsymbol{R})$.

In particular, $\hat{A}^\pm(\boldsymbol{R}) = (1/\sqrt{2})(\pm\partial/\partial x + W(x, \boldsymbol{R}))$ in the one-dimensional case, where $W(x, \boldsymbol{R})$ is the superpotential and must satisfy the nonlinear differential equations [24]:

$$W^2(x, \boldsymbol{R}) + \frac{\partial}{\partial x} W(x, \boldsymbol{R}) = 2V_1(x, \boldsymbol{R}) \tag{2.4a}$$

and

$$W^2(x, \boldsymbol{R}) - \frac{\partial}{\partial x} W(x, \boldsymbol{R}) = 2V_2(x, \boldsymbol{R}). \tag{2.4b}$$

In view of (2.1), supersymmetry is preserved at any instant. Further, we demand that once a positive energy eigenstate of $\hat{H}_1$ (say $|n_1(\boldsymbol{R}_0)\rangle$) is initially supersymmetrically related with a positive energy eigenstate of $\hat{H}_2$ (say $|n_2(\boldsymbol{R}_0)\rangle$), then they will be supersymmetrically related throughout the evolution such that

$$\hat{A}^-(\boldsymbol{R})|n_1(\boldsymbol{R})\rangle = \sqrt{E_n(\boldsymbol{R})}|n_2(\boldsymbol{R})\rangle \tag{2.5a}$$

$$\hat{A}^+(\boldsymbol{R})|n_2(\boldsymbol{R})\rangle = \sqrt{E_n(\boldsymbol{R})}|n_1(\boldsymbol{R})\rangle \tag{2.5b}$$

where $\boldsymbol{R}$ is the value of the parameters at time t and $E_n(\boldsymbol{R})$ (>0) is the common energy eigenvalue of $|n_1(\boldsymbol{R})\rangle$ and $|n_2(\boldsymbol{R})\rangle$.

According to Witten's paper [23], our requirement may be satisfied in the one-dimensional case by conjugation transformation. A supersymmetric quantum system at a particular $\boldsymbol{R}$ is related to the initial system (say at $\boldsymbol{R}_0$) by conjugation transformation:

$$\hat{Q}^+(\boldsymbol{R}) = (\exp(-F(x, \boldsymbol{R})))\hat{Q}^+(\boldsymbol{R}_0)(\exp F(x, \boldsymbol{R})) \tag{2.6}$$

$$\hat{Q}(\boldsymbol{R}) = (\exp F(x, \boldsymbol{R}))\hat{Q}(\boldsymbol{R}_0)(\exp(-F(x, \boldsymbol{R}))) \tag{2.7}$$

$$\hat{\mathscr{H}}(\boldsymbol{R}) = \{\hat{Q}^+(\boldsymbol{R}), \hat{Q}(\boldsymbol{R})\} \tag{2.8}$$

where $F(x, \boldsymbol{R})$ is a function that satisfies $\partial F/\partial x = W(x, \boldsymbol{R}) - W(x, \boldsymbol{R}_0)$.

As discussed in [23], the numbers of zero-energy ground states of $\hat{H}_1$ and $\hat{H}_2$ are separately invariant under conjugation transformation as long as the asymptotic behaviour of $W(x, \boldsymbol{R})$ at large $|x|$ is unchanged. Therefore, the 'structures' of the spectra of $\hat{H}_1$ and $\hat{H}_2$ are unchanged throughout the transformation and our requirement is satisfied.

In higher-dimensional cases, conjugation transformation may be different from (2.6)-(2.8) and the argument of asymptotic behaviour of superpotential may need to be modified. Anyway, in the following discussions, we assume that (2.5) are satisfied during the evolution of the system no matter whether the evolution is conjugation transformation.

3. Difference between Berry's phases

The supersymmetrically related Hamiltonians $\hat{H}_1$ and $\hat{H}_2$ described in the previous section are separately governing two different quantum systems. The Berry's phases of these two systems under the adiabatic evolution, with positive energy initial states $|n_1(\boldsymbol{R}_0)\rangle$ and $|n_2(\boldsymbol{R}_0)\rangle$ which are supersymmetrically related (so satisfy (2.5)), along a circuit $\mathscr{C}$ in the parameter space are given by [1]:

$$\gamma_1(\mathscr{C}) = \mathrm{i}\oint_{\mathscr{C}}\langle n_1(\boldsymbol{R})|\nabla_R n_1(\boldsymbol{R})\rangle \cdot \mathrm{d}\boldsymbol{R} \tag{3.1a}$$

$$\gamma_2(\mathscr{C}) = \mathrm{i}\oint_{\mathscr{C}}\langle n_2(\boldsymbol{R})|\nabla_R n_2(\boldsymbol{R})\rangle \cdot \mathrm{d}\boldsymbol{R}. \tag{3.1b}$$

In (3.1), the states $|n_1(\boldsymbol{R})\rangle$ and $|n_2(\boldsymbol{R})\rangle$ are the instantaneous positive energy eigenstates of $\hat{H}_1(\boldsymbol{R})$ and $\hat{H}_2(\boldsymbol{R})$ respectively and can be chosen to be related as (2.5) for the parameters $\boldsymbol{R}$ on the circuit $\mathscr{C}$.

It should be noted that we further assume, as done in [1], that $|n_1(\boldsymbol{R})\rangle$ and $|n_2(\boldsymbol{R})\rangle$ are discrete and single-valued in the parameter domain that includes the circuit $\mathscr{C}$.

We first consider the operation of ∇_R on both sides of (2.5) arriving at:

$$(\nabla_R\hat{A}^-(\boldsymbol{R}))|n_1(\boldsymbol{R})\rangle + \hat{A}^-(\boldsymbol{R})|\nabla_R n_1(\boldsymbol{R})\rangle = (\nabla_R\sqrt{E_n(\boldsymbol{R})})|n_2(\boldsymbol{R})\rangle + \sqrt{E_n(\boldsymbol{R})}|\nabla_R n_2(\boldsymbol{R})\rangle \tag{3.2a}$$

$$(\nabla_R \hat{A}^+(\boldsymbol{R}))|n_2(\boldsymbol{R})\rangle + \hat{A}^+(\boldsymbol{R})|\nabla_R n_2(\boldsymbol{R})\rangle$$
$$= (\nabla_R \sqrt{E_n(\boldsymbol{R})})|n_1(\boldsymbol{R})\rangle + \sqrt{E_n(\boldsymbol{R})}|\nabla_R n_1(\boldsymbol{R})\rangle. \tag{3.2b}$$

We then attach these two equations by the arbitrary positive energy eigenbras $\langle m_2(\boldsymbol{R})|$ and $\langle m_1(\boldsymbol{R})|$ respectively:

$$\langle m_2(\boldsymbol{R})|(\nabla_R \hat{A}^-(\boldsymbol{R}))|n_1(\boldsymbol{R})\rangle + \sqrt{E_m(\boldsymbol{R})}\langle m_1(\boldsymbol{R})|\nabla_R n_1(\boldsymbol{R})\rangle$$
$$= \nabla_R \sqrt{E_n(\boldsymbol{R})}\delta_{mn} + \sqrt{E_n(\boldsymbol{R})}\langle m_2(\boldsymbol{R})|\nabla_R n_2(\boldsymbol{R})\rangle \tag{3.3a}$$
$$\langle m_1(\boldsymbol{R})|(\nabla_R A^+(\boldsymbol{R}))|n_2(\boldsymbol{R})\rangle + \sqrt{E_m(\boldsymbol{R})}\langle m_2(\boldsymbol{R})|\nabla_R n_2(\boldsymbol{R})\rangle$$
$$= \nabla_R \sqrt{E_n(\boldsymbol{R})}\delta_{mn} + \sqrt{E_n(\boldsymbol{R})}\langle m_1(\boldsymbol{R})|\nabla_R n_1(\boldsymbol{R})\rangle. \tag{3.3b}$$

In particular, by using only $\langle n_2(\boldsymbol{R})|$ and $\langle n_1(\boldsymbol{R})|$ for calculation of Berry's phases, we have:

$$\langle n_2(\boldsymbol{R})|(\nabla_R \hat{A}^-(\boldsymbol{R}))|n_1(\boldsymbol{R})\rangle + \sqrt{E_n(\boldsymbol{R})}\langle n_1(\boldsymbol{R})|\nabla_R n_1(\boldsymbol{R})\rangle$$
$$= \nabla_R \sqrt{E_n(\boldsymbol{R})} + \sqrt{E_n(\boldsymbol{R})}\langle n_2(\boldsymbol{R})|\nabla_R n_2(\boldsymbol{R})\rangle \tag{3.4a}$$
$$\langle n_1(\boldsymbol{R})|(\nabla_R \hat{A}^+(\boldsymbol{R}))|n_2(\boldsymbol{R})\rangle + \sqrt{E_n(\boldsymbol{R})}\langle n_2(\boldsymbol{R})|\nabla_R n_2(\boldsymbol{R})\rangle$$
$$= \nabla_R \sqrt{E_n(\boldsymbol{R})} + \sqrt{E_n(\boldsymbol{R})}\langle n_1(\boldsymbol{R})|\nabla_R n_1(\boldsymbol{R})\rangle. \tag{3.4b}$$

It is found that the integrands in (3.1) appear in (3.4) and we can then obtain the difference between them:

$$\Delta(\boldsymbol{R}) \equiv \langle n_1(\boldsymbol{R})|\nabla_R n_1(\boldsymbol{R})\rangle - \langle n_2(\boldsymbol{R})|\nabla_R n_2(\boldsymbol{R})\rangle$$
$$= (\nabla_R \sqrt{E_n(\boldsymbol{R})} - \langle n_2(\boldsymbol{R})|(\nabla_R \hat{A}^-(\boldsymbol{R}))|n_1(\boldsymbol{R})\rangle)/\sqrt{E_n(\boldsymbol{R})} \tag{3.5a}$$

by (3.4*a*), or

$$(\langle n_1(\boldsymbol{R})|(\nabla_R \hat{A}^+(\boldsymbol{R}))|n_2(\boldsymbol{R})\rangle - \nabla_R \sqrt{E_n(\boldsymbol{R})})/\sqrt{E_n(\boldsymbol{R})} \tag{3.5b}$$

by (3.4*b*).

It is obvious that $\langle n_2(\boldsymbol{R})|(\nabla_R \hat{A}^-(\boldsymbol{R}))|n_1(\boldsymbol{R})\rangle$ is in fact the complex conjugate of $\langle n_1(\boldsymbol{R})|(\nabla_R \hat{A}^+(\boldsymbol{R}))|n_2(\boldsymbol{R})\rangle$. Moreover, the difference $\Delta(\boldsymbol{R})$ is purely imaginary because both integrands are purely imaginary and so we have from (3.5*b*):

$$\mathrm{Re}\langle n_1(\boldsymbol{R})|(\nabla_R \hat{A}^+(\boldsymbol{R}))|n_2(\boldsymbol{R})\rangle/\sqrt{E_n(\boldsymbol{R})} = (\nabla_R \sqrt{E_n(\boldsymbol{R})})/\sqrt{E_n(\boldsymbol{R})} \tag{3.6}$$

which is divergence of some function of $\boldsymbol{R}$ and will vanish under circuit integration.

Finally, using (3.5)-(3.6) and the conjugate relation of $\langle n_2(\boldsymbol{R})|(\nabla_R \hat{A}^-(\boldsymbol{R}))|n_1(\boldsymbol{R})\rangle$ and $\langle n_1(\boldsymbol{R})|(\nabla_R \hat{A}^+(\boldsymbol{R}))|n_2(\boldsymbol{R})\rangle$, the difference in integrands $\Delta(\boldsymbol{R})$ is then given by:

$$\Delta(\boldsymbol{R}) = \mathrm{i}\ \mathrm{Im}\langle n_1(\boldsymbol{R})|(\nabla_R \hat{A}^+(\boldsymbol{R}))|n_2(\boldsymbol{R})\rangle/\sqrt{E_n(\boldsymbol{R})} \tag{3.7}$$

and the difference of the Berry's phases can be obtained:

$$\gamma_1(\mathscr{C}) - \gamma_2(\mathscr{C}) = \mathrm{i}\oint_{\mathscr{C}} \Delta(\boldsymbol{R}) \cdot \mathrm{d}\boldsymbol{R}. \tag{3.8}$$

We thus derived an explicit form for the difference of the Berry's phases for the two supersymmetrically related quantum systems.

In particular, if $\Delta(\boldsymbol{R}) = \nabla_R \mathscr{F}(\boldsymbol{R})$ for some function $\mathscr{F}(\boldsymbol{R})$, $\langle n_1(\boldsymbol{R})|\nabla_R n_1(\boldsymbol{R})\rangle$ and $\langle n_2(\boldsymbol{R})|\nabla_R n_2(\boldsymbol{R})\rangle$ will be related by gauge transformation and the Berry's phases acquired in the two supersymmetrically related quantum systems under adiabatic evolution along a circuit $\mathscr{C}$ will be equal.

4. The invariant topological quantity

Using the fact that $\langle n_2(\boldsymbol{R})|(\nabla_R\hat{A}^-(\boldsymbol{R}))|n_1(\boldsymbol{R})\rangle$ is the complex conjugate of $\langle n_1(\boldsymbol{R})|(\nabla_R\hat{A}^+(\boldsymbol{R}))|n_2(\boldsymbol{R})\rangle$, we can rewrite (3.7) as:

$$\begin{aligned}\langle n_1(\boldsymbol{R})|\nabla_R n_1(\boldsymbol{R})\rangle-\langle n_2(\boldsymbol{R})|\nabla_R n_2(\boldsymbol{R})\rangle\\ &=\mathrm{i}\,\mathrm{Im}\langle n_1(\boldsymbol{R})|(\nabla_R\hat{A}^+(\boldsymbol{R}))|n_2(\boldsymbol{R})\rangle/\sqrt{E_n(\boldsymbol{R})}\\ &=(1/2\sqrt{E_n(\boldsymbol{R})})(\langle n_1(\boldsymbol{R})|(\nabla_R\hat{A}^+(\boldsymbol{R}))|n_2(\boldsymbol{R})\rangle\\ &\quad-\langle n_2(\boldsymbol{R})|(\nabla_R\hat{A}^-(\boldsymbol{R}))|n_1(\boldsymbol{R})\rangle).\end{aligned}\tag{4.1}$$

After some rearrangements and in view of (2.5), we arrive at an equality:

$$\begin{aligned}\langle n_1(\boldsymbol{R})|\nabla_R n_1(\boldsymbol{R})\rangle+\mathrm{i}\,\mathrm{Im}\langle n_1(\boldsymbol{R})|\hat{A}^+(\boldsymbol{R})(\nabla_R\hat{A}^-(\boldsymbol{R}))|n_1(\boldsymbol{R})\rangle|n_1(\boldsymbol{R})\rangle/(2E_n(\boldsymbol{R}))\\ =\langle n_2(\boldsymbol{R})|\nabla_R n_2(\boldsymbol{R})\rangle+\mathrm{i}\,\mathrm{Im}\langle n_2(\boldsymbol{R})|\hat{A}^-(\boldsymbol{R})(\nabla_R\hat{A}^+(\boldsymbol{R}))|n_2(\boldsymbol{R})\rangle/(2E_n(\boldsymbol{R})).\end{aligned}\tag{4.2}$$

In the equality (4.2), we observe that the quantities on both sides do not have the same form. However, because the roles of the two Hamiltonians can be exchanged, we may rewrite $\hat{H}_2(\boldsymbol{R})$ in a form similar to that of $\hat{H}_1(\boldsymbol{R})$ in (2.3) and vice versa, that is:

$$\hat{H}_2(\boldsymbol{R})=\hat{B}^+(\boldsymbol{R})\hat{B}^-(\boldsymbol{R})\tag{4.3a}$$

$$\hat{H}_1(\boldsymbol{R})=\hat{B}^-(\boldsymbol{R})\hat{B}^+(\boldsymbol{R})\tag{4.3b}$$

where $\hat{B}^\pm(\boldsymbol{R})$ are some linear operators different from $\hat{A}^\pm(\boldsymbol{R})$.

Based on (4.3), we can still obtain an equality similar to (4.2) but with $\hat{A}^\pm(\boldsymbol{R})$ replaced by $\hat{B}^\mp(\boldsymbol{R})$:

$$\begin{aligned}\langle n_1(\boldsymbol{R})|\nabla_R n_1(\boldsymbol{R})\rangle+\mathrm{i}\,\mathrm{Im}\langle n_1(\boldsymbol{R})|\hat{B}^-(\boldsymbol{R})(\nabla_R\hat{B}^+(\boldsymbol{R}))|n_1(\boldsymbol{R})\rangle/(2E_n(\boldsymbol{R}))\\ &=\langle n_2(\boldsymbol{R})|\nabla_R n_2(\boldsymbol{R})\rangle\\ &\quad+\mathrm{i}\,\mathrm{Im}\langle n_2(\boldsymbol{R})|\hat{B}^+(\boldsymbol{R})(\nabla_R\hat{B}^-(\boldsymbol{R}))|n_2(\boldsymbol{R})\rangle/(2E_n(\boldsymbol{R})).\end{aligned}\tag{4.4}$$

With the help of (3.6) and (3.7), it is elementary to prove that:

$$\langle n_1(\boldsymbol{R})|\hat{B}^-(\boldsymbol{R})(\nabla_R\hat{B}^+(\boldsymbol{R}))|n_1(\boldsymbol{R})\rangle=\langle n_1(\boldsymbol{R})|\hat{A}^+(\boldsymbol{R})(\nabla_R\hat{A}^-(\boldsymbol{R}))|n_1(\boldsymbol{R})\rangle\tag{4.5a}$$

$$\langle n_2(\boldsymbol{R})|\hat{B}^+(\boldsymbol{R})(\nabla_R\hat{B}^-(\boldsymbol{R}))|n_2(\boldsymbol{R})\rangle=\langle n_2(\boldsymbol{R})|\hat{A}^-(\boldsymbol{R})(\nabla_R\hat{A}^+(\boldsymbol{R}))|n_2(\boldsymbol{R})\rangle.\tag{4.5b}$$

Therefore, by combining (4.2) and (4.4), the equality becomes:

$$\begin{aligned}\langle n_1(\boldsymbol{R})|\nabla_R n_1(\boldsymbol{R})\rangle+\mathrm{i}\,\mathrm{Im}\langle n_1(\boldsymbol{R})|\hat{A}^+(\boldsymbol{R})(\nabla_R\hat{A}^-(\boldsymbol{R}))|n_1(\boldsymbol{R})\rangle/(2E_n(\boldsymbol{R}))\\ &=\langle n_2(\boldsymbol{R})|\nabla_R n_2(\boldsymbol{R})\rangle\\ &\quad+\mathrm{i}\,\mathrm{Im}\langle n_2(\boldsymbol{R})|\hat{B}^+(\boldsymbol{R})(\nabla_R\hat{B}^-(\boldsymbol{R}))|n_2(\boldsymbol{R})\rangle/(2E_n(\boldsymbol{R})).\end{aligned}\tag{4.6}$$

Clearly, both sides in (4.6) share the same form and we can then conclude that the topological quantity, which depends on the circuit $\mathscr{C}$,

$$\mathrm{i}\oint_{\mathscr{C}}\sigma_i(\boldsymbol{R})\cdot\mathrm{d}\boldsymbol{R}=\mathrm{i}\oint_{\mathscr{C}}(\alpha_i(\boldsymbol{R})+\beta_i(\boldsymbol{R}))\cdot\mathrm{d}\boldsymbol{R}\tag{4.7}$$

is invariant in the two supersymmetrically related systems whose Hamiltonians are written in the forms: $\hat{H}_i = \hat{A}_i^+ \hat{A}_i^-$, where $i = 1$ or 2. In the above, $\alpha_i(\boldsymbol{R})$ and $\beta_i(\boldsymbol{R})$ are given by:

$$\alpha_i(\boldsymbol{R}) = \langle n_i(\boldsymbol{R}) | \nabla_R n_i(\boldsymbol{R}) \rangle \tag{4.8}$$

$$\beta_i(\boldsymbol{R}) = \mathrm{i}\ \mathrm{Im} \langle n_i(\boldsymbol{R}) | \hat{A}_i^+(\boldsymbol{R})(\nabla_R \hat{A}_i^-(\boldsymbol{R})) | n_i(\boldsymbol{R}) \rangle / (2E_n(\boldsymbol{R})) \tag{4.9}$$

$$\sigma_i(\boldsymbol{R}) = \alpha_i(\boldsymbol{R}) + \beta_i(\boldsymbol{R}). \tag{4.10}$$

We now consider the replacement of $|n_i(\boldsymbol{R})\rangle$ by $\exp\{\mathrm{i}\mu(\boldsymbol{R})\}|n_i(\boldsymbol{R})\rangle$ and find that the invariant quantities in (4.8)–(4.10) are transformed as:

$$\alpha_i(\boldsymbol{R}) \to \alpha_i(\boldsymbol{R}) + \mathrm{i}\nabla_R \mu(\boldsymbol{R}) \tag{4.11}$$

$$\beta_i(\boldsymbol{R}) \to \beta_i(\boldsymbol{R}) \tag{4.12}$$

$$\sigma_i(\boldsymbol{R}) \to \sigma_i(\boldsymbol{R}) + \mathrm{i}\nabla_R \mu(\boldsymbol{R}). \tag{4.13}$$

It is noticed that $\alpha_i(\boldsymbol{R})$ and $\sigma_i(\boldsymbol{R})$ transform as gauge potentials and hence can be regarded to be connections defined in the Hermitian line bundles ($\alpha_i(\boldsymbol{R})$ is identified with the natural connections as discussion in [3]) associated with the quantum systems under consideration. Since the equality stated in (4.6), $\sigma_i(\boldsymbol{R})$ is an invariant connection in the two supersymmetrically related systems and the phase-like invariant quantity expressed in (4.7) is the holonomy corresponding to such connection.

5. Examples

5.1. The Aharonov–Bohm effect

We would like to consider an example of the Aharonov–Bohm effect [36] as discussed in [1]. Let there be a particle with unit charge confined to a box situated at $\boldsymbol{R}$ and not penetrated by the flux line as shown in figure 1 in [1]. We write the Hamiltonian of this system as:

$$\hat{H}_1(\boldsymbol{R}) = \tfrac{1}{2}(\hat{\boldsymbol{p}} - \boldsymbol{a}(\boldsymbol{r}))^2 + V_1(\boldsymbol{r} - \boldsymbol{R}), \tag{5.1.1}$$

where $\boldsymbol{a}(\boldsymbol{r})$ is the vector potential, $V_1(\boldsymbol{r} - \boldsymbol{R})$ is the potential experienced by the particle inside the box and hence depends on the related position vector $\boldsymbol{r} - \boldsymbol{R}$.

As stated in [1], the energy eigenstate $|n_1(\boldsymbol{R})\rangle$ of $\hat{H}_1(\boldsymbol{R})$ with positive energy eigenvalue E_n (independent of $\boldsymbol{R}$) is given by:

$$\langle \boldsymbol{r} | n_1(\boldsymbol{R}) \rangle = \exp\left\{ \mathrm{i} \int_R^r \mathrm{d}\boldsymbol{r}' \cdot \boldsymbol{a}(\boldsymbol{r}') \right\} \psi_n(\boldsymbol{r} - \boldsymbol{R}) \tag{5.1.2}$$

where $\psi_n(\boldsymbol{r} - \boldsymbol{R})$ is the energy eigenwavefunction of $\hat{H}_1(\boldsymbol{R})$ with energy eigenvalue E_n as the flux is absent ($\boldsymbol{a}(\boldsymbol{r}) = 0$).

The Hamiltonian $\hat{H}_1(\boldsymbol{R})$ can be factorized into the form (2.3):

$$\begin{aligned} \hat{H}_1(\boldsymbol{R}) &= \tfrac{1}{2}(\nabla - \mathrm{i}\boldsymbol{a}(\boldsymbol{r}) + w(\boldsymbol{r} - \boldsymbol{R}))(-\nabla + \mathrm{i}\boldsymbol{a}(\boldsymbol{r}) + w(\boldsymbol{r} - \boldsymbol{R})) \\ &= \hat{A}^+(\boldsymbol{R})\hat{A}^-(\boldsymbol{R}) \end{aligned} \tag{5.1.3}$$

in which we use $\hat{\boldsymbol{p}} = -\mathrm{i}\nabla$ (set $\hbar = 1$) and $w(\boldsymbol{r} - \boldsymbol{R})$ is the superpotential which satisfies the differential equation:

$$2V_1(\boldsymbol{r} - \boldsymbol{R}) = w^2(\boldsymbol{r} - \boldsymbol{R}) + \nabla w(\boldsymbol{r} - \boldsymbol{R}). \tag{5.1.4}$$

Moreover the expressions of operators $\hat{A}^{\pm}(\boldsymbol{R})$ are then given by:

$$\hat{A}^{\pm}(\boldsymbol{R}) = (1/\sqrt{2})(\pm\nabla \mp \mathrm{i}\boldsymbol{a}(\boldsymbol{r}) + w(\boldsymbol{r}-\boldsymbol{R})). \tag{5.1.5}$$

We then construct the supersymmetric partner (denoted by $\hat{H}_2(\boldsymbol{R})$) for $\hat{H}_1(\boldsymbol{R})$:

$$\begin{aligned}\hat{H}_2(\boldsymbol{R}) &= \hat{A}^{-}(\boldsymbol{R})\hat{A}^{+}(\boldsymbol{R}) \\ &= \tfrac{1}{2}(\hat{\boldsymbol{p}} - \boldsymbol{a}(\boldsymbol{r}))^2 + V_2(\boldsymbol{r}-\boldsymbol{R})\end{aligned} \tag{5.1.6}$$

where $V_2(\boldsymbol{r}-\boldsymbol{R})$ is given by:

$$2V_2(\boldsymbol{r}-\boldsymbol{R}) = w^2(\boldsymbol{r}-\boldsymbol{R}) - \nabla w(\boldsymbol{r}-\boldsymbol{R}). \tag{5.1.7}$$

The energy eigenstate, denoted by $|n_2(\boldsymbol{R})\rangle$, can be obtained by the operation of $\hat{A}^{-}(\boldsymbol{R})$ on $|n_1(\boldsymbol{R})\rangle$ as stated in (2.5):

$$|n_2(\boldsymbol{R})\rangle = (1/\sqrt{E_n})\hat{A}^{-}(\boldsymbol{R})|n_1(\boldsymbol{R})\rangle \tag{5.1.8}$$

where $1/\sqrt{E_n}$ is the normalization factor. Moreover, due to the form of $\hat{H}_2(\boldsymbol{R})$ (5.1.6), $|n_2(\boldsymbol{R})\rangle$ can also be expressed similarly to (5.1.2):

$$\langle \boldsymbol{r}|n_2(\boldsymbol{R})\rangle = \exp\left\{\mathrm{i}\int_R^r \mathrm{d}\boldsymbol{r}'\boldsymbol{a}(\boldsymbol{r}')\right\}\phi_n(\boldsymbol{r}-\boldsymbol{R}) \tag{5.1.9}$$

where $\phi_n(\boldsymbol{r}-\boldsymbol{R})$ is the energy eigenwavefunction of $\hat{H}_2(\boldsymbol{R})$ with energy eigenvalue E_n as the flux is absent ($\boldsymbol{a}(\boldsymbol{r})=0$) and satisfies

$$\phi_n(\boldsymbol{r}-\boldsymbol{R}) = (1/\sqrt{E_n})(-\nabla + w(\boldsymbol{r}-\boldsymbol{R}))\psi_n(\boldsymbol{r}-\boldsymbol{R}). \tag{5.1.10}$$

As stated in [1], the quantities $\alpha_i(\boldsymbol{R})$ ($i=1$ or 2) are given by:

$$\begin{aligned}\alpha_1(\boldsymbol{R}) &= \langle n_1(\boldsymbol{R})|\nabla_R n_1(\boldsymbol{R})\rangle \\ &= \iiint \mathrm{d}^3\boldsymbol{r}\psi_n^*(\boldsymbol{r}-\boldsymbol{R})\{-\mathrm{i}\boldsymbol{a}(\boldsymbol{R})\psi_n(\boldsymbol{r}-\boldsymbol{R}) + \nabla_R\psi_n(\boldsymbol{r}-\boldsymbol{R})\} \\ &= -\mathrm{i}\boldsymbol{a}(\boldsymbol{R})\end{aligned} \tag{5.1.11a}$$

and

$$\begin{aligned}\alpha_2(\boldsymbol{R}) &= \langle n_2(\boldsymbol{R})|\nabla_R n_2(\boldsymbol{R})\rangle \\ &= \iiint \mathrm{d}^3\boldsymbol{r}\phi_n^*(\boldsymbol{r}-\boldsymbol{R})\{-\mathrm{i}\boldsymbol{a}(\boldsymbol{R})\phi_n(\boldsymbol{r}-\boldsymbol{R}) + \nabla_R\phi_n(\boldsymbol{r}-\boldsymbol{R})\} \\ &= -\mathrm{i}\boldsymbol{a}(\boldsymbol{R}).\end{aligned} \tag{5.1.11b}$$

We see that $\alpha_1(\boldsymbol{R})$ is equal to $\alpha_2(\boldsymbol{R})$ and depend on the vector potential $\boldsymbol{a}(\boldsymbol{R})$ only. The difference $\Delta(\boldsymbol{R})$ (defined in (3.5)) is then identical to zero and the Berry's phases associated with the two systems are given by:

$$\gamma_1(\boldsymbol{R}) = \gamma_2(\boldsymbol{R}) = \oint_{\mathscr{C}} \boldsymbol{a}(\boldsymbol{R})\,\mathrm{d}\boldsymbol{R}. \tag{5.1.12}$$

Finally, the invariant quantities (the holonomies defined in (4.7)) of the two systems are identical to the Berry's phases in this particular example.

5.2. Spin-$\frac{1}{2}$ particle in magnetic field

Let us consider a spin-$\frac{1}{2}$ particle in a time varying magnetic field $\boldsymbol{B}$ with a background field $\frac{1}{2}$:

$$\hat{H}_1 = \mu \boldsymbol{B} \cdot \hat{S} + \tfrac{1}{2} \tag{5.2.1}$$

where $\boldsymbol{B} = |\boldsymbol{B}|(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$.

In which, θ and ϕ are the usual spherical polar angles which are time dependent. For simplicity, we assume the magnitude of the magnetic field is constant and equal to 1, such that $|\boldsymbol{B}| = 1$. Therefore, the Hamiltonian in (5.2.1) is modified to be:

$$\hat{H}_1 = \boldsymbol{n} \cdot \hat{S} + \tfrac{1}{2} \tag{5.2.2}$$

where $\boldsymbol{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$.

In (5.2.2) we have taken $\mu = \hbar = 1$ for convenience. It is noticed that $\hat{H}_1$ is parametrized by the spherical polar angles (θ, ϕ) and hence denoted by $\hat{H}_1(\theta, \phi)$. The parameter space is then simply 2-sphere.

We can rewrite $\hat{H}_1(\theta, \phi)$ in a form similar to (2.3) by introducing a set of instantaneous spin operators:

$$\hat{S}_x(\theta, \phi) = \hat{S}_x \sin\phi - \hat{S}_y \cos\phi \tag{5.2.3a}$$

$$\hat{S}_y(\theta, \phi) = \hat{S}_x \cos\theta\cos\phi + \hat{S}_y \cos\theta\sin\phi - \hat{S}_z \sin\theta \tag{5.2.3b}$$

$$\hat{S}_z(\theta, \phi) = \hat{S}_x \sin\theta\cos\phi + \hat{S}_y \sin\theta\sin\phi + \hat{S}_z \cos\theta \tag{5.2.3c}$$

and the instantaneous spin ladder operators are defined by:

$$\hat{S}'_+(\theta, \phi) = \hat{S}'_x(\theta, \phi) + \mathrm{i}\hat{S}'_y(\theta, \phi) \tag{5.2.4a}$$

$$\hat{S}'_-(\theta, \phi) = \hat{S}'_x(\theta, \phi) - \mathrm{i}\hat{S}'_y(\theta, \phi). \tag{5.2.4b}$$

Then the Hamiltonian $\hat{H}_1(\theta, \phi)$ becomes:

$$\begin{aligned} \hat{H}_1(\theta, \phi) &= \tfrac{1}{2} + \hat{S}'_z(\theta, \phi) \\ &= \hat{S}'_+(\theta, \phi)\hat{S}'_-(\theta, \phi) \end{aligned} \tag{5.2.5}$$

and its supersymmetric partner $\hat{H}_2(\theta, \phi)$ is given by:

$$\begin{aligned} \hat{H}_2(\theta, \phi) &= \hat{S}'_-(\theta, \phi)\hat{S}'_+(\theta, \phi) \\ &= \tfrac{1}{2} - \hat{S}'_z(\theta, \phi). \end{aligned} \tag{5.2.6}$$

Obviously, $\hat{H}_2(\theta, \phi)$ has the same form as that of $\hat{H}_1(\theta, \phi)$ but with the magnetic field in the opposite direction. At any instant, $\hat{H}_1(\theta, \phi)$ and $\hat{H}_2(\theta, \phi)$ share the same two-dimensional Hilbert space. We may assume that the instantaneous Hilbert space is spanned by the instantaneous eigenstates of $\hat{H}_1(\theta, \phi)$ denoted by $|\Psi_+(\theta, \phi)\rangle$ and $|\Psi_-(\theta, \phi)\rangle$ such that:

$$\hat{H}_1(\theta, \phi)|\Psi_+(\theta, \phi)\rangle = |\Psi_+(\theta, \phi)\rangle \tag{5.2.7a}$$

$$\hat{H}_1(\theta, \phi)|\Psi_-(\theta, \phi)\rangle = 0 \tag{5.2.7b}$$

or more precisely,

$$\hat{S}'_z(\theta, \phi)|\Psi_+(\theta, \phi)\rangle = \tfrac{1}{2}|\Psi_+(\theta, \phi)\rangle \tag{5.2.8a}$$

$$\hat{S}'_z(\theta, \phi)|\Psi_-(\theta, \phi)\rangle = -\tfrac{1}{2}|\Psi_-(\theta, \phi)\rangle. \tag{5.2.8b}$$

Once $|\Psi_+(\theta,\phi)\rangle$ is given, $|\Psi_-(\theta,\phi)\rangle$ can be obtained by the instantaneous spin ladder operator defined by (5.2.4*b*):

$$\hat{S}'_-(\theta,\phi)|\Psi_+(\theta,\phi)\rangle=|\Psi_-(\theta,\phi)\rangle \tag{5.2.9a}$$

and conversely, by (5.2.4*a*):

$$\hat{S}'_+(\theta,\phi)|\Psi_-(\theta,\phi)\rangle=|\Psi_+(\theta,\phi)\rangle. \tag{5.2.9b}$$

Obviously, the states $|\Psi_\pm(\theta,\phi)\rangle$ are also eigenstates of $\hat{H}_2(\theta,\phi)$ with eigenvalues 0 and 1 respectively:

$$\hat{H}_2(\theta,\phi)|\Psi_-(\theta,\phi)\rangle=|\Psi_-(\theta,\phi)\rangle \tag{5.2.10a}$$

$$\hat{H}_2(\theta,\phi)|\Psi_+(\theta,\phi)\rangle=0. \tag{5.2.10b}$$

Moreover, in view of (5.2.5) and (5.2.6), $\hat{S}'_\pm(\theta,\phi)$ are acting as the supersymmetric ladder operators. So that the states $|\Psi_\pm(\theta,\phi)\rangle$ are supersymmetrically related states in accordance with (5.2.9).

One possible expression for $|\Psi_+(\theta,\phi)\rangle$ is given by:

$$|\Psi_+(\theta,\phi)\rangle=\begin{pmatrix}\cos(\theta/2)\\ e^{i\phi}\sin(\theta/2)\end{pmatrix} \tag{5.2.11a}$$

and its supersymmetrically related state is:

$$|\Psi_-(\theta,\phi)\rangle=i\begin{pmatrix}\sin(\theta/2)\\ -e^{i\phi}\cos(\theta/2)\end{pmatrix}. \tag{5.2.11b}$$

We further assume that the evolutions of the systems are adiabatic so that we can apply our arguments in the previous sections for this example.

We can derive the element $\alpha_1\cdot d\boldsymbol{R}$, with $\boldsymbol{R}=(\theta,\phi)$ and $\nabla_R=(\partial/\partial\theta,\partial/\partial\phi)$, by using the expression (5.2.11*a*):

$$\alpha_1\cdot d\boldsymbol{R}=\langle\Psi_+(\theta,\phi)|\nabla_R\Psi_+(\theta,\phi)\rangle\cdot d\boldsymbol{R}=(i/2)(1-\cos\theta)\,d\phi. \tag{5.2.12}$$

Moreover, we can also evaluate the element $\beta_1\cdot d\boldsymbol{R}$:

$$\begin{aligned}\beta_1\cdot d\boldsymbol{R}&=\langle\Psi_+(\theta,\phi)|S'_+(\theta,\phi)(\nabla_R S'_-(\theta,\phi))|\Psi_+(\theta,\phi)\rangle/2\cdot d\boldsymbol{R}\\ &=\langle\Psi_-(\theta,\phi)|(\nabla_R S'_-(\theta,\phi))|\Psi_+(\theta,\phi)\rangle/2\cdot d\boldsymbol{R}\\ &=(i/2)\cos\theta\,d\phi\end{aligned} \tag{5.2.13}$$

which combines with (5.2.12) to give the invariant connection obtained in section 4, relations (4.8)-(4.10), i.e.:

$$\sigma_1\cdot d\boldsymbol{R}=(\alpha_1+\beta_1)\cdot d\boldsymbol{R}=(i/2)\,d\phi. \tag{5.2.14}$$

The corresponding connection form for $\hat{H}_2(\theta,\phi)$, $\sigma_2\cdot d\boldsymbol{R}$, is also $(i/2)\,d\phi$ because it is invariant in the two systems according to (4.6). Moreover, because the curvature form corresponding to $\sigma_i\cdot d\boldsymbol{R}$ vanishes identically, the connection form $\sigma_i\cdot d\boldsymbol{R}$ is called a flat connection [37]. Furthermore $\alpha_2\cdot d\boldsymbol{R}$ can be calculated by using (5.2.13) and (5.2.14) and the fact that $\beta_1=\beta_2^*$:

$$\begin{aligned}\alpha_2\cdot d\boldsymbol{R}&=(i/2)\,d\phi-\beta_2\cdot d\boldsymbol{R}\\ &=(i/2)\,d\phi+\beta_1\cdot d\boldsymbol{R}=(i/2)(1+\cos\theta)\,d\phi.\end{aligned} \tag{5.2.15}$$

Certainly, $\alpha_2\cdot d\boldsymbol{R}$ can be calculated directly as well as $\alpha_1\cdot d\boldsymbol{R}$ in (5.2.12).

By the way, the Berry's phases of the supersymmetrically related systems, their difference and the path dependent phase-like quantity, $\mathrm{i}\oint_{\mathscr{C}}\sigma_i\cdot\mathrm{d}\boldsymbol{R}$, can be found by using (5.2.12)-(5.2.15):

$$\gamma_1(\mathscr{C})=\mathrm{i}\oint_{\mathscr{C}}\cdot\alpha_1\,\mathrm{d}\boldsymbol{R}=-\gamma_2(\mathscr{C})=-\mathrm{i}\oint_{\mathscr{C}}\alpha_2\cdot\mathrm{d}\boldsymbol{R}=(-1/2)\Omega(\mathscr{C}) \qquad (5.2.16)$$

$$\gamma_1(\mathscr{C})-\gamma_2(\mathscr{C})=-\Omega(\mathscr{C}) \qquad (5.2.17)$$

$$\mathrm{i}\oint_{\mathscr{C}}\sigma_i\cdot\mathrm{d}\boldsymbol{R}=0 \qquad (5.2.18)$$

where $\Omega(\mathscr{C})$ is the solid angle that $\mathscr{C}$ subtends at $\boldsymbol{B}=0$ (the origin of 2-sphere).

Furthermore, we would like to explain the physical meaning of the invariant connection form $\sigma_i\cdot\mathrm{d}\boldsymbol{R}$ $(i=1,2)$ in our example. It appears that the issue is related to the difficulty in defining a spinor on a 2-sphere. The expression of positive energy eigenstate of $\hat{H}_1(\theta,\phi)$ in (5.2.11*a*) is reasonable everywhere except at the south pole. Therefore, it is essential to have a complementary expression for the same state as expressed in (5.2.11*a*) [38]:

$$|\Phi_+(\theta,\phi)\rangle=\begin{pmatrix}\mathrm{e}^{-\mathrm{i}\phi}\cos(\theta/2)\\ \sin(\theta/2)\end{pmatrix}. \qquad (5.2.19)$$

It is easy to see that $|\Phi_+(\theta,\phi)\rangle$ is reasonable everywhere except at the north pole and differs from $|\Psi_+(\theta,\phi)\rangle$ by a phase factor:

$$|\Phi_+(\theta,\phi)\rangle=\mathrm{e}^{\mathrm{i}\phi}|\Psi_+(\theta,\phi)\rangle. \qquad (5.2.20)$$

The element $\alpha_1\cdot\mathrm{d}\boldsymbol{R}$ calculated by using (5.2.19) or (5.2.20) is different from that calculated by using (5.2.11*a*) and corresponds to the different choices of connections $(\mathscr{A}_+)$ on the eigenspace bundle [38-40], such that:

$$\mathrm{i}\mathscr{A}_+(\Psi)=\langle\Psi_+(\theta,\phi)|\nabla_R\Psi_+(\theta,\phi)\rangle\cdot\mathrm{d}\boldsymbol{R}=(\mathrm{i}/2)(1-\cos\theta)\,\mathrm{d}\phi \qquad (5.2.21a)$$

$$\mathrm{i}\mathscr{A}_+(\Phi)=\langle\Phi_+(\theta,\phi)|\nabla_R\Phi_+(\theta,\phi)\rangle\cdot\mathrm{d}\boldsymbol{R}=(\mathrm{i}/2)(-1-\cos\theta)\,\mathrm{d}\phi. \qquad (5.2.21b)$$

Clearly, $\mathscr{A}_+(\Psi)$ and $\mathscr{A}_+(\Phi)$ differ by a gauge transformation,

$$\mathscr{A}_+(\Psi)=\mathscr{A}_+(\Phi)+\mathrm{d}\phi. \qquad (5.2.22)$$

Similar arguments can be applied to the positive energy eigenstate of $\hat{H}_2(\theta,\phi)$ and introducing a complementary state, i.e. $|\Phi_-(\theta,\phi)\rangle$,

$$|\Phi_-(\theta,\phi)\rangle=\mathrm{e}^{\mathrm{i}\phi}|\Psi_-(\theta,\phi)\rangle=\mathrm{i}\begin{pmatrix}\mathrm{e}^{-\mathrm{i}\phi}\sin(\theta/2)\\ -\cos(\theta/2)\end{pmatrix}. \qquad (5.2.23)$$

which is analogous to (5.2.19).

Also we can obtain a gauge transformation analogous to (5.2.22) by using (5.2.11*b*):

$$\mathscr{A}_-(\Psi)=\mathscr{A}_-(\Phi)-\mathrm{d}\phi. \qquad (5.2.24)$$

Finally, we can conclude, based on (5.2.14), (5.2.22) and (5.2.24), that the invariant connection form, $\sigma_i\cdot\mathrm{d}\boldsymbol{R}$, is corresponding to the essential gauge transformation (up to a factor i/2) shared by the two supersymmetrically related systems in this example.

6. Discussions

We have obtained in section 3 the difference of the Berry's phases in the two supersymmetrically related systems, as given in (3.8). In section 4, we have successfully constructed a topological quantity ($i\oint_{\mathscr{C}} \sigma_i(\boldsymbol{R})\cdot d\boldsymbol{R}$) which is path dependent, phase-like, including the Berry's phase and, more important, this quantity is invariant in the two supersymmetrically related systems. Such a topological quantity can be understood as the holonomy corresponding to the invariant connection ($\sigma_i(\boldsymbol{R})$) we constructed.

Our aim is to show the invariance of the topological quantity in two supersymmetrically related quantum systems. In the process of such analysis, we find that by introducing the linear operators $\hat{B}^{\pm}(\boldsymbol{R})$ as defined in (4.3), we can rewrite the equality (4.2) into the desirable equality (4.6). However, by only considering (4.3a), we have the freedom to choose another pair of the operators $\hat{B}^{\pm}(\boldsymbol{R})$ and hence the topological quantity in (4.7) depends on the choice of $\hat{B}^{\pm}(\boldsymbol{R})$ in general. On the other hand, we must emphasize that by taking constraint (4.3b) into account, $\hat{B}^{\pm}(\boldsymbol{R})$ are uniquely defined without any ambiguity.

It is also worth mentioning that the dynamical phases acquired by the two supersymmetrically related quantum systems under adiabatic evolution, as discussed in section 2, are identical and equal to $\oint_{\mathscr{C}} E_n(\boldsymbol{R})\cdot d\boldsymbol{R}$ while the Berry's phases are not. On the other hand, the topological quantities we obtained can in some sense be regarded as 'topological phases' which are invariant for supersymmetrically related quantum systems just like the dynamical phases. It would be an interesting analysis in future to find out whether this topological quantity has any physical implications.

In section 5, we presented two examples. In the Aharonov-Bohm example, the difference of the Berry's phases of the quantum systems under considerationn are identical to zero and the topological invariant quantity is nothing but identical to the Berry's phase of the system. We have also taken the system of spin-$\frac{1}{2}$ particle in magnetic field as an example in our analysis. We have evaluated the Berry's phases as in (5.2.16), the difference of the Berry's phases in (5.2.17) and the invariant topological quantity as expressed in (5.2.18) of the supersymmetrically related systems. Moreover, we have shown that the invariant quantity $\sigma_i\cdot d\boldsymbol{R}$ in this example corresponds to the essential gauge transformation [38–40] shared by the two supersymmetrically related systems. Whether such a correspondence is valid in other systems appears to be worth pursuing and awaits for further investigations.

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